JOURNAL OF APPROXIMATION THEORY 53, 350–353 (1988)

Convergence of the Derivatives of Lagrange Interpolating Polynomials Based on the Roots of Hermite Polynomials

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Communicated by R. Bojanic

Received February 3, 1986

In this paper we consider Lagrange interpolation on the real axis. We use the zeros of the classical Hermite polynomials as nodes of interpolation. Freud [4] and Nevai [7, 8] have proved that Lagrange interpolation based on Hermite abscissas produces a convergent approximation process in $(-\infty, \infty)$ for a wide class of functions.

The aim of the present paper is to show that the derivatives of the Lagrange interpolating polynomials based on the roots of Hermite polynomials converge to the derivatives of the interpolated function.

The Hermite polynomials of degree n are

$$H_n(x) = (-1)^n e^{x^2}\{e^{-x^2}\}^{(n)}, \qquad n = 1, 2, ...$$

It is well known that Hermite polynomials are orthogonal on the real line with respect to the weight function e^{-x^2}. We will use the zeros

$$x_{1n} < x_{2n} < \cdots < x_{nn}$$

of $H_n(x)$ as nodes of the Lagrange interpolation. If there is no danger of misunderstanding we will write x_k instead of x_{kn}. The Hermite zeros are symmetrical, that is,

$$x_k = -x_{n-k+1}, \qquad k = 1, 2, ..., n.$$

If n is odd, then

$$x_{(n+1)/2} = 0.$$

We will need the following important relation for the largest zero of $H_n(x)$:

$$x_n = -x_1 \sim n^{1/2},$$

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(Szegő [6, 6.31.19]). Here we use the symbol $\sim$ as in Szegő [6, p. 1]: if sequences z_n, w_n of numbers have the property that all $w_n \neq 0$ and $0 \leqslant \underline{\lim}\, |z_n/w_n| \leqslant \overline{\lim}\, |z_n/w_n| < \infty$, then we write $z_n \sim w_n$.

Now let f be a function defined on the real line. We denote its Lagrange interpolating polynomial based on the Hermite abscissas by

$$L_n(f; x) = \sum_{k=1}^{n} f(x_k)\, l_{kn}(x), \qquad n = 1, 2, ...,$$

where

$$l_{kn}(x) = \frac{H_n(x)}{(x - x_k)\, H'(x_k)}, \qquad k = 1, 2, ..., n.$$

To measure the continuity of an arbitrary function g we apply the usual modulus of continuity

$$\omega(g; \delta) = \sup_{|t| \leqslant \delta} |g(x+t) - g(t)|.$$

In what follows $f^{(0)}$ will denote f.

THEOREM. *Suppose that for some integer $r \geqslant 0$, $f^{(r)}$ exists and is uniformly continuous on $(-\infty, \infty)$. Then*

$$|f^{(i)}(x) - L_n^{(i)}(f; x)| = O(1)\, \omega(f^{(r)}; n^{-1/2})\, n^{-r/2+i}(\log n + e^{x^2/2})$$

for $|x| \leqslant x_n$, $i = 0, 1, 2, ..., r$; $O(1)$ is independent of x and n.

COROLLARY. *If $\lim_{\delta \to 0+} \omega(f^{(r)}; \delta) \log \delta = 0$ then*

$$\lim_{n \to \infty} L_n^{(i)}(f; x) = f^{(i)}(x)$$

for $-\infty < x < \infty$ and $i = 0, 1, 2, ..., [r/2]$ (the integer part of $r/2$). The convergence is uniform in every finite interval. The hypothesis of the corollary holds, e.g., if $f^{(r)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$.

We mention that Freud [4] has given a similar estimate in the case $i = 0$. To prove our theorem we need some lemmas.

LEMMA 1. *If $f^{(r)}$ exists and is continuous for some $r \geqslant 0$, furthermore let $A > 0$, then there exists a polynomial G_n of degree $n \geqslant 4r + 5$ at most, that*

$$|f^{(i)}(x) - G_n^{(i)}(f; x)| = O(1)\, \omega\left(f^{(r)}; \frac{\sqrt{A^2 - x^2}}{n}\right)\left(\frac{\sqrt{A^2 - x^2}}{n}\right)^{r-i}$$

for $|x| \leqslant A$, $i = 0, 1, 2, ..., r$; $O(1)$ depends only i.

COROLLARY. *If* $A = x_n \sim n^{1/2}$, *then*

$$|f^{(i)}(x) - G_n^{(i)}(f; x)| = O(1)\,\omega(f^{(r)}; n^{-1/2})\, n^{-r/2 + i/2}$$

for $|x| \leqslant x_n$ *and* $i = 0, 1, ..., r$.

Proof. The lemma is an easy consequence of Gopengauz's theorem [5].

LEMMA 2 (Freud [4, Theorem 1]). *We have for all real* x *and* $n = 1, 2, ...,$

$$\sum_{k=1}^{n} |l_{kn}(x)| = O(1)(\log n + e^{x^2/2}).$$

LEMMA 3 (Bernstein [3]). *Let* $P_m(x)$ *be a polynomial of degree* m *and* $B > 0$, *then we have*

$$|P_m^{(i)}(x)| \leqslant \left(\frac{i}{B^2 - x^2}\right)^{i/2} m^i \max_{|t| \leqslant B} |P_m(t)|$$

if $|x| < B$ *and* $i = 1, 2, ..., m$.

Proof of Theorem. First we consider the case $0 \leqslant x \leqslant x_n$. Let $G_{n-1}(f; x)$ be a polynomial defined in Lemma 1, where the degree of G_{n-1} is $n-1$ at most, $n - 1 \geqslant 4r + 5$. Then we have by the corollary of Lemma 1,

$$\begin{aligned}
&|f^{(i)}(x) - L_n^{(i)}(f; x)| \\
&\quad \leqslant |f^{(i)}(x) - G_{n-1}^{(i)}(f; x)| + |G_{n-1}^{(i)}(f; x) - L_n^{(i)}(f; x)| \\
&\quad = O(1)\,\omega(f^{(r)}; n^{-1/2})\, n^{-r/2 + i/2} + |L_n^{(i)}(G_{n-1} f - f; x)|.
\end{aligned}$$

Applying Lemma 3 to the second summand, with $P_m(x) = L_n(G_{n-1} f - f; x)$ and $B = x + \varepsilon$ ($0 < \varepsilon \leqslant 1$ will be chosen later), we get

$$\begin{aligned}
|f^{(i)}(x) - L_n^{(i)}(f; x)| &= O(1)(f^{(r)}; n^{-1/2})\, n^{-r/2 + i/2} + O(1)\left(\frac{1}{2\varepsilon x + \varepsilon^2}\right)^{i/2} \\
&\quad \times n^i \max_{|t| \leqslant x + \varepsilon} \left| \sum_{k=1}^{n} [G_{n-1}(f; x_k) - f(x_k)]\, l_{kn}(t) \right|.
\end{aligned}$$

Using the corollary of Lemma 1 and Lemma 2, we can bound the second summand by

$$O(1)\frac{1}{(2\varepsilon x + \varepsilon^2)^{i/2}}\,\omega(f^{(r)}; n^{-1/2})\, n^{-r/2 + i}(\log n + \max_{|t| \leqslant x + \varepsilon} e^{t^2/2}).$$

Choosing $\varepsilon = 1$ if $0 \leqslant x \leqslant 1$, and $\varepsilon = 1/x$ if $1 < x \leqslant x_n$, we obtain for our expression

$$O(1)\,\omega(f^{(r)}; n^{-1/2})\, n^{-r/2+i}(\log n + e^{x^2/2})$$

which proves the theorem if $0 \leqslant x \leqslant x_n$.

In the case $-x_n \leqslant x < 0$ $(-x_n = x_1)$, we introduce the notation $f^-(x) = f(-x)$. A simple calculation yields $L_n(f^-; -x) = L_n(f; x)$. Since $0 < -x \leqslant x_n$ and $\omega(f^{-(r)}; \delta) = \omega(f^{(r)}; \delta)$, we may write

$$|f^{(i)}(x) - L_n^{(i)}(f; x)| = O(1)\,\omega(f^{(r)}; n^{-1/2})\, n^{-r/2+i}(\log n + e^{x^2/2}),$$

if $-x_n \leqslant x < 0$.

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